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Two Approaches to Non-Commutative Geometry*

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Abstract

Looking to the history of mathematics one could find out two outer approaches to Geometry. First one (algebraic) is due to DESCARTES and second one (group-theoretic)—to KLEIN. We will see that they are not rivalling but are tied (by GALOIS). We also examine their modern life as philosophies of non-commutative geometry. Connections between different objects (see *keywords*) are discussed.

Contents

1   Introduction

> Descartes was the first great modern philosopher, a
> founder of modern biology, a first-rate physicist, and
> only incidentally a mathematician.
> M. Kline [29, § 15.3].

If one takes a look on mathematics as a collection of facts about a large diversity of objects covered by the Mathematic Subject Classification by digits from 00 till 93, then it will be difficult to explain why we are referring to mathematics as a "united and inseparable" science. Even the common origin of all mathematical fields laid somewhere in *Elements* of Euclid hardly be an excuse. Indeed western philosophy and psychology as well commonly rooted in works of Great Greeks (Plato or Aristotle?), but it will be offensive for a philosopher as well as for a psychologist do not distinguish them nowadays. In contrast, in mathematics the most exiting and welcome results usually combine facts, notions and ideas of very remote fields (the already classical example is Atiah-Singer theorem linking algebra, analysis and geometry).

So even while exponentially growing mathematical facts remain linked by some underlying ideas, which are essentially vital for the mathematical enterprise. Thus it seems worth enough for a mathematician to think (and sometime to write) not only about *new* mathematical facts but also about *old* original ideas. Particularly relations between new facts and old ideas deserve special attention.

In this paper we return to two great ideas familiar to every mathematician: *Coordinate method* of R. DESCARTES and *Erlangen program* of F. KLEIN. They specify two different approaches to geometry and become actual now in connections with such an interesting and fascinate area of research as non-commutative geometry [12, 36].

The paper format is as follows. After some preliminary remarks in Sections 2–4 we will arrive to paper's core in Section 5: it is an abstract scheme of analytic function theories. We will interpret classic examples accordingly to the presented scheme in Section 6. In Section 7 we will expand consideration to non-commutative spaces with physical application presented in Section 8. For a reader primary interested in analytic function theory Section 5 and 6 form an independent reading.

A feature of the paper is a healthy portion of self-irony. It will not be an abuse if a reader will smile during the reading as the author did during the writing.

2   Coordinates: from Descartes to Nowadays

> ... coordinate geometry changed the face of mathematics.
>
> M. Kline [29, § 15.6].

The first danger of a deep split in mathematics condensed before 17-th century. The split shown up between synthetic geometry of Greeks and abstract algebra of Arabs. It was overcome by works of FERMAT and DESCARTES on analytic geometry. Descartes pointed out that geometric construction call for adding, subtracting, multiplying, and dividing segments of lines. But this is exactly four operations of algebra. So one could express and solve geometric problems in algebraic terms. The crucial steps are:

1. To introduce a coordinate system, which allows us to label geometrical points by sets of numbers or *coordinates*. Of course, coordinates are *functions* of points.

2. To link desired geometrical properties and problems with *equations* of coordinate. Then solutions to equations will give solutions of original geometric problems.

Four algebraic operations suggest to consider coordinates from Step 1 as members of a commutative algebra of functions of geometric points. Moreover, it seems unavoidable to use other functions from this algebra to construct and solve equations from Step 2. In such a way the commutative algebra of functions become the central object of new analytic geometry. From geometry this algebra inherits some additional structures (metrical, topological, etc.)

Having a vocabulary that allows us to translate from the geometrical language to algebraic one nothing could prevent one to use it in the opposite direction: *what are geometrical counterparts of such and such algebraic notions?* Particularly, what will happen if we start from an arbitrary commutative algebra (with a relevant additional structure) instead coordinate one? The answer to a question of this type is given by the celebrated theorem of I.M. GELFAND and M.A. NAIMARK [24, § 4.3]:

Theorem 2.1 (Gelfand-Naimark) *Any commutative Banach algebra $\mathfrak{A}$ could be realized as the algebra $C(X)$ of all continuous functions over a topological space X. The points of X are labelled by the maximal ideals (or characters) of $\mathfrak{A}$.*

So looking for adventures and new experience in geometry one should try to begin from *non-commutative* algebras. This gives the name of resulting theory as *non-commutative geometry*. Very attractive (and in some sense archetypical) object is *anticommutative* coordinates, which satisfy to identity $x_1 x_2 = -x_2 x_1$ instead of commutation rule $x_1 x_2 = x_2 x_1$:

> In recent years, several attempts have been made to develop anticommutative analogs of geometric notions. Most of these attempts are rooted in one of the great ideas of mathematics in this century, the idea that geometry is recovered from algebra by taking prime or maximal ideals of a commutative ring. Thus one replaces a commutative ring by an exterior algebra and one tries to do something similar. [8]

On this road one could found natural unification of vector geometry and Grassmann algebras.

There are still many other opportunities. As it always happens with generalizations of a non-trivial notion, there is no *the* non-commutative geometry but there *are* non-commutative geometries (for example, [12] and [36]).

REMARK 2.2 It is important that all described constructions deserve the name of *geometry* not only by peculiar rules of the game named mathematics

Commutative - Classic	**Non-Commutative - Quantum**
Complex variables	Operator in Hilbert space
Real variables	Selfadjoint operators
Infinitesimal variable	Compact operator
Infinitesimal of order α	$\mu_n(T) = O(n^{-\alpha})$
Integral	$\int T = \log \text{trace } T$

Table 1: Vocabulary between Commutative-Classic and Non-Commutative-Quantum Geometries. Here $\mu_n(T)$ is stand for n-th eigenvalue of a compact operator T.

but also by their relations with the real word. Their ambitions are to describe the structure of actual space(-time) on the quantum level (see Section 8) exactly as classic geometry succeed it for the Nile valley. Thus as commutative geometry is the language of classic mechanics so non-commutative geometry is supposed to be a language for quantum physics. So in modern science non-commutative and quantum are usually synonymous.

Example 2.3 Quantum mechanics begun from the Heisenberg commutation relations $XY = YX + i\hbar I$ and was a source of inspiration for functional analysis. Thus an additional structure on non-commutative algebra, which describes geometrical objects, comes often from operator theory. Table 1 contains a vocabulary for non-commutative geometry taken from [7], which is devoted to description of quantized theory of gravitational field.

On this way one could find such exiting examples as a curvature or a connection on a geometric set consisting of two points only [12]. We will discuss such a vocabulary with more details in Section 8.

3   Klein, Galois, Descartes

> ... not all of geometry can be incorporated in Klein's scheme.
>
> M. Kline [29, § 38.5].

The second (group-theoretic) viewpoint on geometry was expressed by F. KLEIN in his famous *Erlangen program* and seems unrelated to coordinate geometry. But this is not true.

Theorem 3.1 *The difference between Klein and Descartes is Galois or*

$$Klein(1872) = Galois(1831) + Descartes(1637) \qquad (3.1)$$

PROOF. The idea of Descartes to reduce geometric problems to the algebraic equations was a brilliant prophecy regarding to the very modest knowledge about algebraic equations in that times. Although his contemporaries could solve already algebraic equations up to fourth degree, i.e., the most general that could be solved in radicals, the general theory of algebraic equation was not even supposed to exist. It is enough to say that for Descartes[1] even negative roots of an equation were false roots not speaking about complex ones [29, §13.5].

We were lucky that another genius, namely GALOIS, leaves to us a message at the end of his very short life: *solvability of equations could be determined via its group of symmetries.* Again we could skip all facts of corresponding theory (for elementary account see for example [29, § 31.4]) and just add it to the coordinate geometry[2]. The result will almost identical with the Erlangen program [29, § 38.5]:

Definition 3.2 Every geometry can be characterized by a *group* (of transformations) and that geometry is really concerned with *invariants* under this group.

To see this achievement in the proper light one should keep in mind that the notion of abstract group was fully recognized only years after Erlangen program [29, § 49.2] (as the notion of a root to an equation was recognized years after Descartes' coordinate geometry). □

Corollary 3.3 *Klein is greater than Descartes:*

$$Klein > Descartes$$

PROOF. Galois is obviously positive ($Galois > 0$), so the (3.1) implies the assertion. □

Problem 3.4 Find a counterexample to the last Corollary.

Hint. See the epigraph.

Even not being universal the Erlangen program offers a way to classify the variety of geometries by means of associated groups [29, § 38.5]. We will link in the Erlangen spirit different analytic function theories and connected functional calculi with group representations. This gives an opportunity to make their classification too.

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[1] We could be surprised how little a genius should be acquainted with mathematical facts to discover the underlying fundamental idea.

[2] Note the appearance of a geometric problem in [29, § 31.5] immediately after the Galois theory in [29, § 31.4]

REMARK 3.5 The connection jokingly presented in this section is often unobserved but is very important. In concrete situations it takes practical forms. Many connections between non-commutative geometry and group representations are mentioned in [42]. For example, an early result on non-commutative geometry—the classification of von Neumann factors of type III—was carried out by means their group of automorphisms [12] (see also Remarks 6.4 and 7.2). And that is also important, such a connection generates a hope that not only particular mathematical facts are linked by fundamental ideas, but also these ideas are connected one with another on a deeper level.

4 The Heisenberg Group and Symmetries of the Manin Plain

> ... I want to popularize the Heisenberg group, which is remarkably little known considering its ubiquity.
> R. Howe [19].

What does Erlangen program tell to a non-commutative geometer? One could do non-commutative geometry considering (quantum) symmetries of non-commutative (=quantum) objects. This approach leads to quantum groups [22, 35, 36]. We will consider a simple example of $M_q(2)$.

We have mentioned already (see Remark 2.3) the Heisenberg commutation relation:

$$[X, Y] = I. \tag{4.1}$$

Here X, Y, and I, viewed as generators of a Lie algebra, produce a Lie group, which is the celebrated Heisenberg (or Weyl) group $\mathbb{H}^1$ [17, 19, 44]. An element $g \in \mathbb{H}^n$ (for any positive integer n) could be represented as $g = (t, z)$ with $t \in \mathbb{R}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and the group law is given by

$$g * g' = (t, z) * (t', z') = (t + t' + \frac{1}{2} \sum_{j=1}^{n} \Im(\bar{z}_j z'_j), z + z'), \tag{4.2}$$

where $\Im z$ denotes the imaginary part of a complex number z. Of course the Heisenberg group is non-commutative and particularly one could find out for $\mathbb{H}^1$:

$$(0, 1) * (0, i) = (1, 0) * (0, i) * (0, 1) \tag{4.3}$$

The last formula is usually called the Weyl commutation relation and is the integrated (or exponentiated) form of (4.1).

Now let us consider a continuous irreducible representation [24, 44] $\pi : \mathbb{H}^1 \to \mathfrak{A}$ of the Heisenberg group in a C^*-algebra $\mathfrak{A}$. Then elements of the

center of $\mathbb{H}^1$, which are of the form $(t,0)$, will map to multipliers ae of the identity $e \in \mathfrak{A}$. For unitary representations we have $|a| = 1$ for all such a.

Let $x = \pi(0,1)$, $y = \pi(0,i)$, $qe = \pi(1,0)$. Then the Weyl commutation relation (4.3) will correspond to

$$xy = qyx \qquad (4.4)$$

The last equation is taken by Yu. Manin [35] as the defining relation for *quantum plane*, which is also often called the Manin plane. As we saw the Manin plane is a representation of the Heisenberg-Weyl group and thus could be named[3] Heisenberg-Weyl-Manin plane. More formal: Manin plane with a parameter $q \in \mathbb{C}$ is the quotient of free algebra generated by elements x and y subject to two-sided ideal generated by the quadratic relation (4.4).

Let us consider the regular representation π_r of $\mathbb{H}^1$ as right shifts on $L_2(\mathbb{H}^1)$ and let $\mathfrak{M}$ be an algebra of operators generated by $x_r = \pi_r(0,1)$ and $y_r = \pi_r(0,i)$. Then Manin plane is a representation π of $\mathfrak{M}$ under which $x = \pi(x_r)$ and $y = \pi(y_r)$. Moreover *an algebraic identity $f(x,y) = 0$ holds on Manin plane for all $q \in \mathbb{C}$ if and only if the algebraic identity $f(x_r, y_r) = 0$ is true in $\mathfrak{M}$.*

Connection between the Manin plane and the Weyl commutation relation was already mentioned in [35] but was not used explicitly. Thus this connection disappeared from the following works. Even fundamental treatise [22] (which has 531 pages!) mentioned the Heisenberg group only twice and in both cases disconnected with the Manin plane. This is especially pity because the Heisenberg group could give new incites for quantum groups, as it does for the Manin plane.

Example 4.1 [4] As it was mentioned at the beginning of the section the quantum group could be viewed as symmetries of quantum object. Particularly $M_q(2)$ is a set of 2×2 matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, "which maps the Manin plane to the Manin plane", namely if x and y satisfy to (4.4) and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

then $x' = ax + by$, $y' = cx + dy$ should again satisfy to (4.4). From this condition for both matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ one could find that

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[3]Now and then the nobility of a mathematical object is measured by the number of its family names.

[4]For brevity the Example is presented very informal. We believe that this is not an abuse.

entries a, b, c, and d of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are subject to the following six identities:

$$ab \;=\; q^{-1}ba, \quad ac = q^{-1}ca, \quad db = qbd, \quad dc = qcd \qquad (4.5)$$
$$bc \;=\; cb, \quad [a,d] = -(q - q^{-1})cb \qquad (4.6)$$

To keep anytime in mind all six identities for any element of $M_q(2)$ is a little bit laborious, isn't it? The Heisenberg group proposes an elegant way out. 2-vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on the Manin plane is of course an element of a representation of $\mathbb{H}^2$ as the Manin plane itself is a representation of $\mathbb{H}^1$. And it is easy to check that due to celebrated Stone-von Neumann theorem [17, 44] six identities (4.5)–(4.6) up to unitary equivalence prescribe that a, b, c, d belong to a representation π_M of $\mathbb{H}^2$ such that:

$$a = \pi_M(0,1,1), \quad b = \pi_M(0,1,i), \quad c = \pi_M(0,i,1), \quad d = \pi_M(0,i,i).$$

Then a product of two elements of $M_q(2)$ is naturally described via representation of $\mathbb{H}^4$ (because entries of different elements pairwise commutes) and so far. Thus $M_q(2)$ *is a representation of* $M_2(\pi_M(\mathbb{H}^\infty))$. There is no problem to think over $\mathbb{H}^\infty$ because consideration is purely algebraic and no topology is involved. Alternatively one could consider $\mathbb{H}^N$ for some unspecified sufficiently large N. Of course, our conclusion about algebraic identities on the Manin plane remains true for $M_q(2)$.

5 Analytic Function Theory from Group Representations

> I learned from I.M. Gel'fand that Mathematics of any kind is a representation theory...
> Yu. I. Manin,
> *Short address on occasion of Professor Gel'fand 70th birthday.*

One could notice that through glasses of Erlangen program the boundary between commutative and non-commutative geometry becomes tiny. Quantum groups became a representation of classic group and the word *non-commutative* is not distinguishing any more. Indeed, the group of symmetries of Euclidean geometry is non-commutative (think on a composition of a shift and a rotation) and a group could act as symmetries of classic and quantum object (symplectic transformations). And even more: the difference between geometry and, say, analysis became unnoticeable.

In this section we are going to proof the following[5]

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[5]We are continuing the definition-theorem-proof style presentation of Section 3.

Theorem 5.1 *Complex analysis = conformal geometry.*

PROOF. The *group* of conformal mapping (particularly for the unit disk they are fraction-linear transformations) plays an important role in any textbook on complex analysis [33, Chap. 10], [38, Chap. 2]. Moreover, two key object of complex analysis, namely the Cauchy-Riemann equations and Cauchy kernel, are *invariants* with respect to these transformations. Thus the Theorem follows from Definition 3.2. □

5.1   Wavelet Transform and Coherent States

We agree with a reader if he/she is not satisfied by the last short proof and would like to see a more detailed account how the core of complex analysis could be reconstructed from representation theory of $SL(2, \mathbb{R})$. We present an abstract scheme, which also could be applied to other analytic function theories [10, 27]. We start from a dry construction followed in the next Section by classic examples, which will justify our usage of personal names.

Let X be a topological space and let G be a group that acts $G : X \to X$ as a transformation $g : x \mapsto g \cdot x$ from the left, i.e., $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. Moreover, let G act on X transitively. Let there exist a measure dx on X such that a representation $\pi(g) : f(x) \mapsto m(g, x)f(g^{-1} \cdot x)$ (with a function $m(g, h)$) is unitary with respect to the scalar product $\langle f_1(x), f_2(x) \rangle_{L_2(X)} = \int_X f_1(x) \bar{f}_2(x) \, d(x)$, i.e.,

$$\langle [\pi(g)f_1](x), [\pi(g)f_2](x) \rangle_{L_2(X)} = \langle f_1(x), f_2(x) \rangle_{L_2(X)} \qquad \forall f_1, f_2 \in L_2(X).$$

We consider the Hilbert space $L_2(X)$ where representation $\pi(g)$ acts by unitary operators.

REMARK 5.2 It is well known that the most developed part of representation theory consider unitary representations in Hilbert spaces. By this reason we restrict our attention to Hilbert spaces of analytic functions, the area usually done by means of the functional analysis technique. We also assume that our functions are complex valued and this is sufficient for examples explicitly considered in the present paper. However the presented scheme is working also for vector valued functions and this is the natural environment for Clifford analysis [6], for example. One also could start from an abstract Hilbert space H with no explicit realization as $L_2(X)$ given.

Let H be a closed compact[6] subgroup of G and let $f_0(x)$ be such a function

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[6]While the compactness will be explicitly used during our abstract consideration, it is not crucial in fact. Example 6.3 will show how one could make a trick for non-compact H.

that H acts on it as the multiplication

$$[\pi(h)f_0](x) = \chi(h)f_0(x) \qquad , \forall h \in H, \tag{5.1}$$

by a function $\chi(h)$, which is a character of H i.e., $f_0(0)$ is a common eigenfunction for all operators $\pi(h)$. Equivalently $f_0(x)$ is a common eigenfunction for operators corresponding under π to a basis of the Lie algebra of H. Note also that $|\chi(h)|^2 = 1$ because π is unitary. $f_0(x)$ is called *vacuum vector* (with respect to subgroup H). We introduce the $F_2(X)$ to be the closed liner subspace of $L_2(X)$ uniquely defined by the conditions:

1. $f_0 \in F_2(X)$;

2. $F_2(X)$ is G-invariant;

3. $F_2(X)$ is G-irreducible.

Thus restriction of π on $F_2(X)$ is an irreducible unitary representation.

The *wavelet transform*[7] $\mathcal{W}$ could be defined for square-integral representations π by the formula [25]

$$\begin{aligned} \mathcal{W} \quad &: \quad F_2(X) \to L_\infty(G) \\ &: \quad f(x) \mapsto \widetilde{f}(g) = \langle f(x), \pi(g)f_0(x)\rangle_{L_2(X)} \end{aligned} \tag{5.2}$$

The principal advantage of the wavelet transform $\mathcal{W}$ is that it express the representation π in geometrical terms. Namely it *intertwins* π and left regular representation λ on G:

$$[\lambda_g \mathcal{W}f](g') = [\mathcal{W}f](g^{-1}g') = \langle f, \pi_{g^{-1}g'}f_0\rangle = \langle \pi_g f, \pi_{g'}f_0\rangle = [\mathcal{W}\pi_g f](g'), \tag{5.3}$$

i.e., $\lambda\mathcal{W} = \mathcal{W}\pi$. Another important feature of W is that it does not lose information, namely function $f(x)$ could *be recovered* as the linear combination of *coherent states* $f_g(x) = [\pi_g f_0](x)$ from its wavelet transform $\widetilde{f}(g)$ [25]:

$$f(x) = \int_G \widetilde{f}(g)f_g(x)\,dg = \int_G \widetilde{f}(g)[\pi_g f_0](x)\,dg, \tag{5.4}$$

where dg is the Haar measure on G normalized such that $\int_G \left|\widetilde{f_0}(g)\right|^2 dg = 1$. One also has an orthogonal *projection* $\widetilde{\mathcal{P}}$ from $L_2(G, dg)$ to image $F_2(G, dg)$

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[7]The subject of coherent states or wavelets have been arising many times in many *applied* areas and the author is not able to give a comprehensive history and proper credits. One could mention important books [13, 28, 37]. We give our references by recent paper [25], where applications to *pure* mathematics were considered.

of $F_2(X)$ under wavelet transform $\mathcal{W}$, which is just a convolution on g with the image $\widetilde{f}_0(g) = \mathcal{W}(f_0(x))$ of the vacuum vector [25]:

$$[\widetilde{\mathcal{P}}w](g') = \int_G w(g)\widetilde{f}_0(g^{-1}g')\,dg. \tag{5.5}$$

5.2 Reduced Wavelets Transform

Our main observation will be that one could be much more economical (if subgroup H is non-trivial) with a help of (5.1): in this case one need to know $\widetilde{f}(g)$ not on the whole group G but only on the homogeneous space G/H [1, § 3].

Let $\Omega = G/H$ and $s : \Omega \to G$ be a continuous mapping [24, § 13.1]. Then any $g \in G$ has a unique decomposition of the form $g = s(a)h$, $a \in \Omega$ and we will write $a = s^{-1}(g)$, $h = r(g) = (s^{-1}(g))^{-1}g$. Note that Ω is a left G-homogeneous space[8] with an action defined in terms of s as follow: $g : a \mapsto s^{-1}(g \cdot s(a))$. Due to (5.1) one could rewrite (5.2) as:

$$
\begin{aligned}
\widetilde{f}(g) &= \langle f(x), \pi(g)f_0(x) \rangle_{L_2(X)} \\
&= \langle f(x), \pi(s(a)h)f_0(x) \rangle_{L_2(X)} \\
&= \langle f(x), \pi(s(a))\pi(h)f_0(x) \rangle_{L_2(X)} \\
&= \langle f(x), \pi(s(a))\chi(h)f_0(x) \rangle_{L_2(X)} \\
&= \bar{\chi}(h)\langle f(x), \pi(s(a))f_0(x) \rangle_{L_2(X)}
\end{aligned}
$$

Thus $\widetilde{f}(g) = \bar{\chi}(h)\widehat{f}(a)$ where

$$\widehat{f}(a) = [\mathcal{C}f](a) = \langle f(x), \pi(s(a))f_0(x) \rangle_{L_2(X)} \tag{5.6}$$

and function $\widetilde{f}(g)$ on G is completely defined by function $\widehat{f}(a)$ on Ω. Formula (5.6) gives us an embedding $\mathcal{C} : F_2(X) \to L_\infty(\Omega)$, which we will call *reduced wavelet transform*. We denote by $F_2(\Omega)$ the image of $\mathcal{C}$ equipped with Hilbert space inner product induced by $\mathcal{C}$ from $F_2(X)$.

Note a special property of $\widetilde{f}_0(g)$ and $\widehat{f}_0(a)$:

$$\widetilde{f}_0(h^{-1}g) = \langle f_0, \pi_{h^{-1}g}f_0 \rangle = \langle \pi_h f_0, \pi_g f_0 \rangle = \langle \chi(h)f_0, \pi_g f_0 \rangle = \chi(h)\widetilde{f}_0(g).$$

It follows from (5.3) that $\mathcal{C}$ intertwines $\rho\mathcal{C} = \mathcal{C}\pi$ representation π with the representation

$$[\rho_g \widehat{f}](a) = \widehat{f}(s^{-1}(g \cdot s(a)))\chi(r(g \cdot s(a))). \tag{5.7}$$

---

[8]Ω with binary operation $(a_1, a_2) \mapsto s^{-1}(s(a_1) \cdot s(a_2))$ becomes a loop of the most general form [39]. Thus theory of reduced wavelet transform developed in this subsection could be considered as *wavelet transform associated with loops*. However we prefer to develop our theory based on groups rather on loops.

While ρ is not completely geometrical as λ in applications it is still more geometrical than original π. In many cases ρ is *representat induced* by the character χ.

If $f_0(x)$ is a vacuum state with respect to H then $f_g(x) = \chi(h)f_{s(a)}(x)$ and we could rewrite (5.4) as follows:

$$\begin{aligned}
f(x) &= \int_G \widetilde{f}(g)f_g(x)\,dg \\
&= \int_\Omega \int_H \widetilde{f}(s(a)h)f_{s(a)h}(x)\,dh\,da \\
&= \int_\Omega \int_H \widehat{f}(a)\bar{\chi}(h)\chi(h)f_{s(a)}(x)\,dh\,da \\
&= \int_\Omega \widehat{f}(a)f_{s(a)}(x)\,da \cdot \int_H |\chi(h)|^2\,dh \\
&= \int_\Omega \widehat{f}(a)f_{s(a)}(x)\,da,
\end{aligned}$$

if the Haar measure dh on H is set in such a way that $\int_H |\chi(h)|^2\,dh = 1$ and $dg = dh\,da$. We define an integral transformation $\mathcal{F}$ according to the last formula:

$$[\mathcal{F}\widehat{f}](x) = \int_\Omega \widehat{f}(a)f_{s(a)}(x)\,da, \qquad (5.8)$$

which has the property $\mathcal{F}\mathcal{C} = I$ on $F_2(X)$ with $\mathcal{C}$ defined in (5.6). One could consider the integral transformation

$$[\mathcal{P}f](x) = [\mathcal{F}\mathcal{C}f](x) = \int_\Omega \langle f(y), f_{s(a)}(y)\rangle_{L_2(X)} f_{s(a)}(x)\,da \qquad (5.9)$$

as defined on whole $L_2(X)$ (not only $F_2(X)$). It is known that $\mathcal{P}$ *is an orthogonal projection* $L_2(X) \to F_2(X)$ [25]. If we formally use linearity of the scalar product $\langle \cdot, \cdot \rangle_{L_2(X)}$ (i.e., assume that the Fubini theorem holds) we could obtain from (5.9)

$$\begin{aligned}
[\mathcal{P}f](x) &= \int_\Omega \langle f(y), f_{s(a)}(y)\rangle_{L_2(X)} f_{s(a)}(x)\,da \\
&= \langle f(y), \int_\Omega f_{s(a)}(y)\bar{f}_{s(a)}(x)\,da\rangle_{L_2(X)} \\
&= \int_X f(y)K(y,x)\,d\mu(y), \qquad (5.10)
\end{aligned}$$

where

$$K(y,x) = \int_\Omega \bar{f}_{s(a)}(y)f_{s(a)}(x)\,da$$

With the "probability $\frac{1}{2}$" (see discussion on the Bergman and the Szegö kernels bellow) the integral (5.10) exists in the standard sense, otherwise it is a singular integral operator (i.e, $K(y,x)$ is a regular function or a distribution).

Sometimes a reduced form $\widehat{\mathcal{P}} : L_2(\Omega) \to F_2(\Omega)$ of the projection $\widetilde{\mathcal{P}}$ (5.5) is of a separate interest. It is an easy calculation that

$$[\widehat{\mathcal{P}}f](a') = \int_\Omega f(a)\widehat{f}_0(s^{-1}(a^{-1} \cdot a'))\bar{\chi}(r(a^{-1} \cdot a'))\, da, \qquad (5.11)$$

where $a^{-1} \cdot a'$ is an informal abbreviation for $(s(a))^{-1} \cdot s(a')$. As we will see its explicit form could be easily calculated in practical cases.

And only at the very end of our consideration we introduce the Taylor series and the Cauchy-Riemann equations. One knows that they are *starting points* in the Weierstrass and the Cauchy approaches to complex analysis correspondingly.

For any decomposition $f_a(x) = \sum_\alpha \psi_\alpha(x)V_\alpha(a)$ of the coherent states $f_a(x)$ by means of functions $V_\alpha(a)$ (where the sum could become eventually an integral) we have the *Taylor series* expansion

$$
\begin{aligned}
\widehat{f}(a) &= \int_X f(x)\bar{f}_a(x)\, dx = \int_X f(x) \sum_\alpha \bar{\psi}_\alpha(x)\bar{V}_\alpha(a)\, dx \\
&= \sum_\alpha \int_X f(x)\bar{\psi}_\alpha(x)\, dx\, \bar{V}_\alpha(a) \\
&= \sum_\alpha^\infty \bar{V}_\alpha(a)f_\alpha, \qquad (5.12)
\end{aligned}
$$

where $f_\alpha = \int_X f(x)\bar{\psi}_\alpha(x)\, dx$. However to be useful within the presented scheme such a decomposition should be connected with structures of G and H. For example, if G is a semisimple Lie group and H its maximal compact subgroup then indices α run through the set of irreducible unitary representations of H, which enter to the representation π of G.

The *Cauchy-Riemann equations* need more discussion. One could observe from (5.3) that the image of $\mathcal{W}$ is invariant under action of the left but right regular representations. Thus $F_2(\Omega)$ is invariant under representation (5.7), which is a pullback of the left regular representation on G, but its right counterpart. Thus generally there is no way to define an action of left-invariant vector fields on Ω, which are infinitesimal generators of right translations, on $L_2(\Omega)$. But there is an exception. Let $\mathfrak{X}_j$ be a maximal set of left-invariant vector fields on G such that

$$\mathfrak{X}_j\widetilde{f}_0(g) = 0.$$

Because $\mathfrak{X}_j$ are left invariant we have $\mathfrak{X}_j \widetilde{f}'_g(g) = 0$ for all g' and thus image of $\mathcal{W}$, which the linear span of $\widetilde{f}'_g(g)$, belongs to intersection of kernels of $\mathfrak{X}_j$. The same remains true if we consider pullback $\widehat{\mathfrak{X}}_j$ of $\mathfrak{X}_j$ to Ω. Note that the number of linearly independent $\widehat{\mathfrak{X}}_j$ is generally less than for $\mathfrak{X}_j$. We call $\widehat{\mathfrak{X}}_j$ as *Cauchy-Riemann-Dirac* operators in connection with their property

$$\widehat{\mathfrak{X}}_j \widehat{f}(g) = 0 \qquad \forall \widehat{f}(g) \in F_2(\Omega). \tag{5.13}$$

Explicit constructions of the Dirac type operator for a discrete series representation could be found in [3, 30].

We do not use Cauchy-Riemann-Dirac operator in our construction, but this does not mean that it is useless. One could found at least such its nice properties:

1. Being a left-invariant operator it naturally encodes an information about symmetry group G.

2. It effectively separates irreducible components of the representation π of G in $L_2(X)$.

3. It has a local nature in a neighborhood of a point vs. transformations, which act globally on the domain.

5.3   Great Names

It is the time to give names of Greats to our formulas. A surprising thing is: all formulas have two names, each one for different circumstances (judgments will be given in Section 6). More precisely we have two alternatives

1. X and Ω are *not* isomorphic as topological spaces with measures. Then $F_2(X)$ becomes the *Hardy space*. The space $L_2(\Omega)$ is *Segal-Bargmann space*. We could consider (5.6) as an abstract analog of the *Cauchy integral formula*, which recovers analytic function $\widehat{f}(a)$ on Ω from its (boundary) values $f(x)$ on X. The formula (5.8) could be named as *Bargamann inverse formula*[9]. The orthogonal projection $\mathcal{P}$ (5.9) is the *Szegö projection* and expression (5.10) is a singular integral operator. In this case of Hardy spaces we could not define a Hilbert space structure on $F_2(\Omega)$ by means of a measure on Ω. Therefor we need to keep both

---

[9]The name is taken from the context of classical Segal-Bargmann space [4, 40] and stand for the mapping from the Segal-Bargmann space to $L_2(\mathbb{R}^n)$. I do not know its name in the context of Hardy space, but it definitely should exist (it is impossible to find out something new for analytic functions in our times).

Notion	Name for case $X \not\sim \Omega$	Name for case $X \sim \Omega$
Space $F_2(X)$	Hardy space	Bergman space
Space $F_2(\Omega)$	Segal-Bargmann space	
Formula (5.6)	Cauchy Integral	Bergman integral
Formula (5.8)	Segal-Bargmann inverse	
Projection (5.10)	Szegö projection	Bergman projection
Projection (5.11)	Segal-Bargmann projection	
Series (5.12)	Taylor series	
Operator (5.13)	Cauchy-Riemann-Dirac operator	

Table 2: Abstract formulas and their classical names. The apparent periodicity of names is not a coincidence or author's arbitrariness.

faces of our theory: the scalar product could be defined by a measure only for $F_2(X)$ and analytic structure is defined via geometry of $F_2(\Omega)$.

2. X and Ω *are* isomorphic as topological spaces with measures. In sharp contrast to the previous case where no name appears twice, this case fully covered by single person, namely BERGMAN. On the other hand, because $X \sim \Omega$ we have twice as less objects in this case. Particularly $F_2(X) \sim F_2(\Omega)$ is known as the *Bergman* space. In this case representation π on $F_2(X) \sim F_2(\Omega)$ due to embedding $s : \Omega \to G$ could be treated as a part of *left regular* representation on $L_2(G)$ by shifts and thus representation π belongs to *discrete series* [34, § VI.5]. The orthogonal projection $\mathcal{P}$ (5.9) is the *Bergman projection* and expression (5.10) is a regular integral operator.

REMARK 5.3 The presented constructions being developed independently has many common points with the theory of harmonic functions on symmetric spaces (see for example [32]). The new feature presented here is the association of function theories with sets of data (G, π, H, f_0) rather symmetric spaces. This allows

1. To distinguish under the common framework the Hardy and different Bergman spaces, which are defined on the same symmetric space.

2. To deal with non-geometrical group action like in the case of the Segal-Bargmann space (see Example 6.3).

3. To apply it not only to semisimple but also nilpotent Lie groups (see Example 6.3).

On the other hand BARGMANN himself realizes that spaces of analytic functions give realizations of representations for semisimple groups and later this idea was used many times (see for example [3], [18, § 5.4], and references therein). We would like emphasize here that not only analytic functions spaces themselves come from representation theory. All principal ingredients (the Cauchy integral formula, the Cauchy-Riemann equations, the Taylor series, etc.) of analytic function theory also appear from such an approach..

6   The Name of the Game

> It's the 'ch' sound in Scottish words like *loch* or German words like *ach*; it's Spanish 'j' and a Russian 'kh'. When you say it correctly to your computer, the terminal may become slightly moist.
> Donald E. Knuth [31, Chap. 1].

It is the great time now to explain personal names appeared in the previous Section. The main example is provided by group $G = SL(2, \mathbb{R})$ (books [20, 34, 44] are our standard references about $SL(2, \mathbb{R})$ and its representations) consisting of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and determinant $ad - bc = 1$. Via identities

$$\alpha = \frac{1}{2}(a + d - ic + ib), \qquad \beta = \frac{1}{2}(c + b - ia + id)$$

we have isomorphism of $SL(2, \mathbb{R})$ with group $SU(1, 1)$ of 2×2 matrices with complex entries of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ such that $|\alpha|^2 - |\beta|^2 = 1$. We will use the last form for $SL(2, \mathbb{R})$ only.

$SL(2, \mathbb{R})$ has the only non-trivial compact closed subgroup H, namely the group of matrix of the form $h_\psi = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$. Now any $g \in SL(2, \mathbb{R})$ has a unique decomposition of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \tag{6.1}$$

where $\psi = \Im \ln \alpha$, $a = \bar{\alpha}^{-1}\beta$, and $|a| < 1$ because $|\alpha|^2 - |\beta|^2 = 1$. Thus we could identify $SL(2, \mathbb{R})/H$ with the unit disk $\mathbb{D}$ and define mapping $s : \mathbb{D} \to SL(2, \mathbb{R})$ as follows

$$s : a \mapsto \frac{1}{\sqrt{1 - |a|^2}} \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}.$$

The invariant measure $d\mu(a)$ on $\mathbb{D}$ coming from decomposition $dg = d\mu(a)\,dh$, where dg and dh are Haar measures on G and H respectively, is equal to

$$d\mu(a) = \frac{da}{\left(1 - |a|^2\right)^2}. \tag{6.2}$$

with da—the standard Lebesgue measure on $\mathbb{D}$.

The formula $g : a \mapsto g \cdot a = s^{-1}(g^{-1}s(a))$ associates with a matrix $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ the fraction-linear transformation of $\mathbb{D}$, which also could be considered as a transformation of $\dot{\mathbb{C}}$ (the one-point compactification of $\mathbb{C}$) of the form

$$g : z \mapsto g \cdot z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \tag{6.3}$$

Complex plane $\mathbb{C}$ is the disjoint union of three orbits, which are acted by (6.3) of G transitively:

1. The unit disk $\mathbb{D} = \{z \mid |z|^2 < 1\}$.

2. The unit circle $\Gamma = \{z \mid |z|^2 = 1\}$.

3. The remain part $\mathbb{C} \setminus \bar{\mathbb{D}} = \{z \mid |z|^2 > 1\}$.

$h_\psi \in H$ acts as the rotation on angle 2ψ and we could identify $H \sim \Gamma$.

Example 6.1 First let us try $X = \Gamma$. We equip X with the standard Lebesgue measure $d\phi$ normalized in such a way that

$$\int_\Gamma |f_0(\phi)|^2 \, d\phi = 1 \text{ with } f_0(\phi) \equiv 1. \tag{6.4}$$

Then there is the unique (up to equivalence) way to make a unitary representation of G in $L_2(X) = L_2(\Gamma, d\phi)$ from (6.3), namely

$$[\pi_g f](e^{i\phi}) = \frac{1}{\bar{\beta} e^{i\phi} + \bar{\alpha}} f\left(\frac{\alpha e^{i\phi} + \beta}{\bar{\beta} e^{i\phi} + \bar{\alpha}}\right). \tag{6.5}$$

This is a realization of the *mock discrete series* of $SL(2, \mathbb{R})$. Function $f_0(e^{i\phi}) \equiv 1$ mentioned in (6.4) transforms as follows

$$[\pi_g f_0](e^{i\phi}) = \frac{1}{\bar{\beta} e^{i\phi} + \bar{\alpha}} \tag{6.6}$$

and particularly has an obvious property $[\pi_{h_\psi} f_0](\phi) = e^{i\psi} f_0(\phi)$, i.e., it is a *vacuum vector* with respect to subgroup H. The smallest linear subspace $F_2(X) \in L_2(X)$ spanned by (6.6) consists of boundaries values of analytic function in the unit disk and is the *Hardy* space. Now reduced wavelet transform (5.6) takes the form

$$
\begin{aligned}
\widehat{f}(a) = [\mathcal{C}f](a) &= \langle f(x), \pi(s(a)) f_0(x) \rangle_{L_2(X)} \\
&= \int_\Gamma f(e^{i\phi}) \frac{\sqrt{1 - |a|^2}}{\bar{a} e^{i\phi} + 1} \, d\phi \\
&= \frac{\sqrt{1 - |a|^2}}{i} \int_\Gamma \frac{f(e^{i\phi})}{a + e^{i\phi}} i e^{i\phi} \, d\phi \\
&= \frac{\sqrt{1 - |a|^2}}{i} \int_\Gamma \frac{f(z)}{a + z} \, dz,
\end{aligned}
\tag{6.7}
$$

where $z = e^{i\phi}$. Of course (6.7) is the *Cauchy integral formula* up to factor $2\pi \sqrt{1 - |a|^2}$. Thus we will write $f(a) = \left(2\pi \sqrt{1 - |a|^2} \right)^{-1} - \widehat{f}(-a)$ for analytic extension of $f(\phi)$ to the unit disk. The factor 2π is due to our normalization (6.4) and $\sqrt{1 - |a|^2}$ is connected with the invariant measure on $\mathbb{D}$.

Consider now a realization of inverse formula (5.8):

$$
\begin{aligned}
f(e^{i\phi}) &= \int_{\mathbb{D}} \widehat{f}(a) \frac{\sqrt{1 - |a|^2}}{\bar{a} e^{i\phi} + 1} \, d\mu(a) \\
&= -\int_{\mathbb{D}} 2\pi \sqrt{1 - |a|^2} f(-a) \frac{\sqrt{1 - |a|^2}}{\bar{a} e^{i\phi} + 1} \frac{da}{(1 - |a|^2)^2} \\
&= 2\pi \int_{\mathbb{D}} \frac{f(a) \, da}{(1 - \bar{a} e^{i\phi})(1 - |a|^2)^2}.
\end{aligned}
\tag{6.8}
$$

The last integral is divergent, the singulyarity is concentrated near the unit circle. A regularization of the integral gives us the *Szëgo* projection:

$$
f(e^{i\phi}) = \pi \int_\Gamma \frac{f(a) \, da}{1 - \bar{a} e^{i\phi}}.
$$

Values of $f(e^{i\phi})$ could be alternatively reconstructed from $f(a)$ by means of the limiting procedure $f(e^{i\phi}) = \lim_{r \to 1} f(r e^{i\phi})$. Formula (6.8) is of minor use in complex analysis but we will need it in Section 7 to construct analytic functional calculus.

Example 6.2 Let us consider second opportunity $X = \mathbb{D}$. For any integer $m \geq 2$ one could select a measure $d\mu_m(w) = 4^{1-m}(1 - |w|^2)^{m-2} dw$, where dw is the standard Lebesgue measure on $\mathbb{D}$, such that action (6.3) could be turn to a unitary representation [34, § IX.3]. Namely

$$[\pi_m(g)f](w) = f\left(\frac{\alpha w + \beta}{\bar{\beta}w + \bar{\alpha}}\right)(\bar{\beta}w + \bar{\alpha})^{-m}. \tag{6.9}$$

If we again select $f_0(w) \equiv 1$ then

$$[\pi_m(g)f_0](w) = (\bar{\beta}w + \bar{\alpha})^{-m},$$

particularly $[\pi_m(f_\psi)f_0](w) = e^{im\psi}f_0(w)$ so this is again a vacuum vector with respect to H. The irreducible subspace $F_2(\mathbb{D})$ generated by $f_0(w)$ consists of analytic functions and is the m-th Bergman space (actually Bergman considered only $m = 2$). Now the transformation (5.6) takes the form

$$
\begin{aligned}
\widehat{f}(a) &= \langle f(w), [\pi_m(s(a))f_0](w)\rangle \\
&= (1 - |a|^2)^{m/2} \int_{\mathbb{D}} \frac{f(w)}{(a\bar{w} + 1)^m} \frac{dw}{(1 - |w|^2)^{2-m}},
\end{aligned}
$$

which is for $m = 2$ the classical Bergman formula up to factor $(1 - |a|^2)^{m/2}$. Note that calculations in standard approaches is "rather lengthy and must be done in stages" [33, § 1.4]. As was mentioned early, its almost identical with its inverse (5.8):

$$
\begin{aligned}
f(w) &= \int_{\mathbb{D}} \widehat{f}(a)f_a(w)\,d\mu(a) = \int_{\mathbb{D}} \frac{\widehat{f}(a)}{(1 + \bar{a}w)^m} \frac{da}{(1 - |a|^2)^2} \\
&= \int_{\mathbb{D}} f(a)\left(\frac{\sqrt{1 - |a|^2}}{1 + \bar{a}w}\right)^m \frac{da}{(1 - |a|^2)^2} \\
&= \int_{\mathbb{D}} \frac{f(a)}{(1 + \bar{a}w)^m} \frac{da}{(1 - |a|^2)^{2-m}},
\end{aligned}
$$

where $f(a) = (1 - |a|^2)^{m/2}\,\widehat{f}(-a)$ is the same function as $f(w)$.

Example 6.3 The purpose of this Example is many-folds. First we would like to justify the Segal-Bargmann name for the formula (5.8). Second we need a demonstration as our scheme will work for

1. The nilpotent Lie group $G = \mathbb{H}^n$.

2. Non-geometric action of the group G.

3. Non-compact subgroup H.

We will consider a representation of the Heisenberg group $\mathbb{H}^n$ (see Section 4) on $L_2(\mathbb{R}^n)$ by operators of shift and multiplication [44, § 1.1]:

$$g = (t, z) : f(x) \rightarrow [\pi_{(t,z)}f](x) = e^{i(2t - \sqrt{2}qx + qp)} f(x - \sqrt{2}p), \qquad z = p + iq, \tag{6.10}$$

i.e., this is the Schrödinger representation with parameter $\hbar = 1$. As a subgroup H we select the center of $\mathbb{H}^n$ consisting of elements $(t, 0)$. It is non-compact but using the special form of representation (6.10) we could consider the cosets[10] $\widetilde{G}$ and $\widetilde{H}$ of G and H by subgroup of elements $(\pi m, 0)$, $m \in \mathbb{Z}$. Then (6.10) also defines a representation of $\widetilde{G}$ and $\widetilde{H} \sim \Gamma$. We consider the Haar measure on $\widetilde{G}$ such that its restriction on $\widetilde{H}$ has the total mass equal to 1.

As "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(x) = e^{-x^2/2}$. Its transformations are defined as follow:

$$\begin{aligned} f_g(x) = [\pi_{(t,z)}f_0](x) &= e^{i(2t - \sqrt{2}qx + qp)} e^{-(x - \sqrt{2}p)^2/2} \\ &= e^{2it - (p^2 + q^2)/2} e^{-((p - iq)^2 + x^2)/2 + \sqrt{2}(p - iq)x} \\ &= e^{2it - z\bar{z}/2} e^{-(\bar{z}^2 + x^2)/2 + \sqrt{2}\bar{z}x}. \end{aligned}$$

Particularly $[\pi_{(t,0)}f_0](x) = e^{-2it} f_0(x)$, i.e., it really is a vacuum vector in the sense of our definition with respect to $\widetilde{H}$. Of course $\Omega = \widetilde{G}/\widetilde{H}$ isomorphic to $\mathbb{C}^n$ and mapping $s : \mathbb{C}^n \rightarrow \widetilde{G}$ simply is defined as $s(z) = (0, z)$. The Haar measure on $\mathbb{H}^n$ is coincide with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [44, § 1.1] thus the invariant measure on Ω also coincide with the Lebesgue measure on $\mathbb{C}^n$. Note also that composition law $s^{-1}(g \cdot s(z))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s^{-1}((s(z_1))^{-1} \cdot s(z_2)) = z_2 - z_1$ and $r((s(z_1))^{-1} \cdot s(z_2)) = \frac{1}{2}\Im \bar{z}_1 z_2$.

Transformation (5.6) takes the form of embedding $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned} \widehat{f}(z) &= \langle f, \pi_{s(z)}f_0 \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}^n} f(x) \, e^{-z\bar{z}/2} \, e^{-(z^2 + x^2)/2 + \sqrt{2}zx} \, dx \end{aligned}$$

---

[10]$\widetilde{G}$ is sometimes called the *reduced* Heisenberg group. It seems that $\widetilde{G}$ is a virtual object, which is important in connection with a selected representation of G.

$$= \quad e^{-z\bar{z}/2}\pi^{-n/4}\int_{\mathbb{R}^n} f(x)\, e^{-(z^2+x^2)/2+\sqrt{2}zx}\, dx, \qquad (6.11)$$

where $z = p + iq$. Then $\widehat{f}(g)$ belongs to $L_2(\mathbb{C}^n, dg)$ or its preferably to say that function $\breve{f}(z) = e^{z\bar{z}/2}\widehat{f}(t_0, z)$ belongs to space $L_2(\mathbb{C}^n, e^{-|z|^2}dg)$ because $\breve{f}(z)$ is analytic in z. Such functions form the *Segal-Bargmann space* [4, 40] $F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ of functions, which are analytic by z and square-integrable with respect the Gaussian measure $e^{-|z|^2}dz$. Analyticity of $\breve{f}(z)$ is equivalent to condition $(\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j I)\widehat{f}(z) = 0$.

The integral in (6.11) is the well known Segal-Bargmann transform [4, 40]. Inverse to it is given by a realization of (5.8):

$$\begin{aligned} f(x) \quad &= \quad \int_{\mathbb{C}^n} \widehat{f}(z) f_{s(z)}(x)\, dz \\ &= \quad \int_{\mathbb{C}^n} \breve{f}(z) e^{-(\bar{z}^2+x^2)/2+\sqrt{2}\bar{z}x}\, e^{-|z|^2}\, dz \qquad (6.12) \end{aligned}$$

and this gives to (5.8) the name of Segal-Bargmann inverse. The corresponding operator $\mathcal{P}$ (5.9) is an identity operator $L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ and (5.9) give an integral presentation of the Dirac delta.

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|z|^2}dg) \to F_2(\mathbb{C}^n, e^{-|z|^2}dg)$ is of interest and is a principal ingredient in Berezin quantization [5, 11]. We could easy find its kernel from (5.11). Indeed, $\widehat{f_0}(z) = e^{-|z|^2}$, then kernel is

$$\begin{aligned} K(z, w) \quad &= \quad \widehat{f_0}(z^{-1} \cdot w)\bar{\chi}(r(z^{-1} \cdot w)) \\ &= \quad \widehat{f_0}(w - z)e^{i\Im(\bar{z}w)} \\ &= \quad \exp(\frac{1}{2}(-|w - z|^2 + w\bar{z} - z\bar{w})) \\ &= \quad \exp(\frac{1}{2}(-|z|^2 - |w|^2) + w\bar{z}). \end{aligned}$$

To receive the reproducing kernel for functions $\breve{f}(z) = e^{|z|^2}\widehat{f}(z)$ in the Segal-Bargmann space we should multiply $K(z, w)$ by $e^{(-|z|^2+|w|^2)/2}$ which gives the standard reproducing kernel $= \exp(-|z|^2 + w\bar{z})$ [4, (1.10)].

REMARK 6.4 Started from the Hardy space H_2 on Γ we recover the unit disk $\mathbb{D}$ in a spirit of the Erlangen program as a homogeneous space connected with its group of symmetry. On the other hand $\mathbb{D}$ carries out a good portion of information about space of maximal ideal of H_∞, which is a subspace of H_2 forming an algebra under multiplication [14, Chap. 6]. So we meet $\mathbb{D}$ if we will look for geometry of H_p from the Descartes viewpoint. This illustrate once more that there is an intimate connection between these two approaches.

REMARK 6.5 The abstract scheme presented here is not a theory with the only example. One could apply it to analytic function theory of several variables. This gives a variety of different function theories both known (see below) and new [10, 27]. For example, the biholomorphic automorphisms [38, Chap. 2] of unit ball in $\mathbb{C}^n$ lead to several complex variable theory, while Möbius transformations [9] of $\mathbb{R}^n$ guide to Clifford analysis [6]. It is interesting that both types of transformations could be represented as fraction-linear ones associated with some 2×2-matrixes and very reminiscent the content of this Section.

On the other hand, historically several complex variable theory was developed as coordinate-wise extension of notion of holomorphy while Clifford analysis took a conceptually different route of a factorization of the Laplacian. It is remarkable that such differently rooted theories could be unified within presented scheme. Moreover, relations between symmetry groups of complex and Clifford analysis explain why one could make conclusions about complex analysis using Clifford technique. This repeats relations between affine and Euclidean geometry, for example.

Considering such a pure mathematic theory as complex analysis we continuously use the language of applications (vacuum vector, coherent states, wavelet transform, etc.). We will consider these applications explicitly later in Section 8.

7 Non-Commutative Conformal Geometry

> The search for generality and unification is one of the
> distinctive features of twentieth century mathematics,
> and functional analysis seeks to achieve these goals.
> M. Kline [29, § 46.1].

Accordingly to F. KLEIN one could associate to every specific geometry a group of symmetries, which characterizes it. This is also true for new geometries discovered in 20-th century. Many books on functional analysis (see for example [14, 23]) contain chapters entitled as *Geometry of Hilbert Spaces* where they present inner product, Pythagorean theorem, orthonormal basis etc. All mentioned notions are invariants of the *group of unitary transformation* of a Hilbert space. On the contrary, the geometry of Hilbert spaces is not directly related to any coordinate algebra. So this is a geometry more in Klein meaning than in Descartes ones.

Considering Banach spaces (or more generally locally convex ones) one has group of continuous *affine* transformations. Its invariant are points, lines, hyperplanes and notions, which could be formulated in their term.

The excellent example is *covexity*. Being continuous these transformations have topological invariants like compactness. Thereafter the Krein-Milman theorem is formulated entirely in terms of continuous affine invariants and thus belong to the body of *infinite dimensional continuous affine geometry*.

Which groups of symmetries produce an interesting geometries in Banach algebras? One could observe that $SL(2, \mathbb{R})$ defines a fraction-linear transformation

$$g : t \mapsto g \cdot t = (\bar{\beta}t + \bar{\alpha})^{-1}(\alpha t + \beta), \qquad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \qquad (7.1)$$

not only for t being a complex number but also any element of a Banach algebra $\mathfrak{A}$ provided $(\bar{\beta}t + \bar{\alpha})^{-1}$ exists. To be sure that (7.1) is defined for any $g \in SL(2, \mathbb{R})$ one could impose the condition $\|t\| < 1$. Let us fix such a $t \in \mathfrak{A}$, $\|t\| \leq 1$. Then elements $g \cdot t$, $g \in SL(2, \mathbb{R})$ form a subset $\mathbb{T}$ of $\mathfrak{A}$ invariant under $SL(2, \mathbb{R})$ and is a homogenous space where $SL(2, \mathbb{R})$ acts by the rule: $g : g' \cdot t \mapsto (gg') \cdot t$. Note also that $\|g \cdot t\| < 1$. Thus it is reasonably to consider $\mathbb{T}$ as an analog of the unit disk $\mathbb{D}$ and try to construct analytic functions on $\mathbb{T}$. This problem is known as a functional calculus of an operator and we would like to present its solution inspired by the Erlangen program.

We are looking for a possibility to assign an element $\Phi(f, s(a) \cdot t) = f(s(a) \cdot t)$ in $\mathfrak{A}$ to every pair $f \in F_2(\Gamma)$ and $s(a) \cdot t \in \mathbb{T}$, $a \in \mathbb{D}$. If we consider $F_2(\Gamma)$ as a linear space then naturally to ask that mapping $\Phi : (f, s(a) \cdot t) \to f(s(a) \cdot t)$ should be linear in f. In line with function theory (6.5) one also could define associated representation of $SL(2, \mathbb{R})$ by the rule:

$$\tau_g \Phi(f, g' \cdot t) = (\bar{\beta}t + \bar{\alpha})^{-1} \Phi(f, (gg') \cdot t). \qquad (7.2)$$

And finally one could assume that (6.5) and (7.2) are in the agreement:

$$\tau_g \Phi(f, g' \cdot t) = \Phi(\pi_g f, g' \cdot t). \qquad (7.3)$$

Thereafter one could obtain the functional calculus $\Phi(f, g \cdot t)$ from a knowledge of $\Phi(f, t)$ for all $f \in F_2(\Gamma)$ using (7.2) and (7.3):

$$\Phi(f, g \cdot t) = (\bar{\beta}t + \bar{\alpha})\Phi(\pi_g f, t).$$

However it is easier to process in another way around. One could define $\Phi(f_0, g \cdot t)$ for a specific f_0 and all g and then extend it to an arbitrary f using (7.3) and linearity. We are now ready to give

Definition 7.1 The *functional calculus* $\Phi(f, g \cdot t)$ for an element t of a Banach algebra $\mathfrak{A}$ is a linear continuous map $\Phi : H_2 \times \mathbb{T} \to \mathfrak{A}$ satisfying to the following conditions

1. Φ intertwines two representations π_g (6.5) and τ_g (7.2) of $SL(2,\mathbb{R})$, namely (7.3) holds.

2. $\Phi(f_0, g \cdot t) = e$ for all g, where $f_0(\phi) \equiv 1$ and e is the unit of $\mathfrak{A}$.

We know (6.8) that any function $f(e^{i\phi})$ has a decomposition

$$f(e^{i\phi}) = 2\pi \int_{\mathbb{D}} \frac{f(a)\,da}{(1 - \bar{a}e^{i\phi})(1 - |a|^2)}.$$

Thus using linearity

$$
\begin{aligned}
\Phi(f(e^{i\phi}), t) &= \Phi\left(2\pi \int_{\mathbb{D}} \frac{f(a)\,da}{(1 - \bar{a}e^{i\phi})(1 - |a|^2)}, t\right) \\
&= 2\pi \int_{\mathbb{D}} f(a)\Phi\left(\frac{1}{1 - \bar{a}e^{i\phi}}, t\right)\frac{da}{(1 - |a|^2)} \quad (7.4) \\
&= -2\pi \int_{\mathbb{D}} f(a)\Phi(\pi_{s(-a)}f_0, t)\frac{da}{1 - |a|^2} \\
&= -2\pi \int_{\mathbb{D}} f(a)\tau_{s(-a)}\Phi(f_0, t)\frac{da}{1 - |a|^2} \quad (7.5) \\
&= 2\pi \int_{\mathbb{D}} f(a)(1 - \bar{a}t)^{-1}\Phi(f_0, s(-a) \cdot t)\frac{da}{1 - |a|^2} \\
&= 2\pi \int_{\mathbb{D}} f(a)(1 - \bar{a}t)^{-1}\frac{da}{1 - |a|^2}. \quad (7.6)
\end{aligned}
$$

Here we use linearity of functional calculus to receive (7.4), its intertwining property to get (7.5), and finally (7.6) that e is always the image of 1. The only difference between the last formula (7.6) and (6.8) is the placing of t instead of $e^{i\phi}$. If one would like to obtain an integral formula for the functional calculus in terms of $f(e^{i\phi})$ itself rather than $f(a)$ then

$$
\begin{aligned}
\Phi(f(e^{i\phi}), t) &= \pi \int_{\mathbb{D}} f(a)(1 - \bar{a}t)^{-1}\frac{da}{1 - |a|^2} \\
&= \pi \int_{\mathbb{D}} \int_{\Gamma} \frac{f(e^{i\phi})\,de^{i\phi}}{a - e^{i\phi}}(1 - \bar{a}t)^{-1}\frac{da}{1 - |a|^2} \\
&= \pi \int_{\Gamma} f(e^{i\phi}) \int_{\mathbb{D}} \frac{(1 - \bar{a}t)^{-1}}{a - e^{i\phi}}\frac{da}{1 - |a|^2}\,de^{i\phi} \\
&= \int_{\Gamma} f(e^{i\phi})(t - e^{i\phi})^{-1}de^{i\phi} \quad (7.7)
\end{aligned}
$$

From our explicit construction it follows that functional calculus is *always exist* and *unique* under assumption $\|t\| < 1$. Moreover from the last formula

it follows also that our functional calculus coincides with the Riesz-Dunford one [15, § VIII.2].

REMARK 7.2 Guided by the Erlangen program we constructed a functional calculus coincided with the classical Dunford-Riesz calculus. The former is traditionally defined in terms of algebra homomorphisms, i.e., on the Descartes geometrical language. So two different approaches again lead to the same answer.

If one compares epigraphs to Sections 5 and 7 then it will be evidently that the functional calculus based on group representations responds to "the search for generality and unification". Besides a funny alternative to classic Riesz-Dunford calculus the given scheme give unified approach to a variety of existing and yet to be discovered calculi [26, Remark 4.2]. We consider some of them in next Section in connections with physical applications.

8   Do We Need That Observables Form an Algebra?

> ... Descartes himself thought that all of physics could
> be reduced to geometry.
>
> M. Kline [29, § 15.6].

We have already mentioned that the development of non-commutative geometry intimately connected with physics, especially with quantum mechanics. Despite the fact that quantum mechanics attracted considerable efforts of physicists and mathematicians during this century we still have not got at hands a consistent quantum theory. Unsolved paradoxes and discrepant interpretations are linked with very elementary quantum models. As a solution to this frustrating situation some physicists adopt a motto that science is permitted to supply inconsistent models "as is" so far they give reasonable numerical predictions to particular experiments[11]. However other researchers still believe that there exists a difference between the standards of scientific enterprise and software selling industry.

We will review some topics in quantization problem in connection with the two approaches to non-commutative geometry. Quantum mechanics begun from experiments clearly indicated that Minkowski four dimensional space-time could not describe microscopical structure of our world (on the other

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[11]And even more strange: there exists a variety of "immortal" theories already contradicting to our experience, see for example the red shift problem in [41].

hand it is also unsuitable for macroscopic description and these two faults became connected [41]). A way out is searched in many directions: dimensionality of microscopical space was raised to $m > 4$ dimensions, its discontinuity was imposed, etc. These search is based on the following conclusion [42]:

> From a quantum mechanical standpoint, the trust of these works was that "space-time" was more logically not a primary concept but one derived from the algebra of "observables" (or operators). In its simplest form the idea was space-time might be, at a deeper level, the spectrum of an appropriate commutative subalgebra that is invariant under the fundamental symmetry group.

The right algebra of quantum observables is often constructed from classic observables by means of *quantization*. A summary of quantization problem could be given as follows (see also [24, § 15.4] and Table 1). The principal objects of classical mechanics in Hamiltonian formalism are:

1. The *phase space*, which is a smooth symplectic manifold M.

2. *Observables* are real functions on M and could be considered as elements of a linear space or algebra.

3. *States* of a physical system are linear functionals on observables.

4. *Dynamics* of observables is defined by a Hamiltonian H and equation

$$\dot{F} = \{H, F\}. \tag{8.1}$$

5. *Symmetries* of the physical system act on observables or states via canonical transformations of M.

Of course points of phase space is in one-to-one correspondence with states of the system and they together are completely defined by the set of observables. The later viewed as an algebra recovers M as its space of maximal ideals. Thus observables, dynamics, and symmetries are primary objects while phase space and states could be restored from them.

The quantum mechanics has its parallel set of notions:

1. *Phase space* is the projective space $P(V)$ of a Hilbert space V.

2. *Observables* are self-adjoint operators on V.

3. *States* of a physical system defined by a vector $\xi \in V$, $\|\xi\| = 1$.

4. *Dynamics* of an observable $\widehat{F}$ defined by a self-adjoint operator $\widehat{H}$ via the Heisenberg equation

$$\dot{\widehat{F}} = \frac{i\hbar}{2\pi}[\widehat{H}, \widehat{F}]. \qquad (8.2)$$

5. *Symmetries* of physical system act on observables or states via unitary operators on V.

Again we could start from an algebra of observables and then realize states and Hilbert space V via GNS construction [24, § 4.3].

The similarity of two descriptions (with correspondence of the Poisson bracket and commutator) makes it very tempting to construct a map $\mathcal{Q}$, which is called the *Dirac quantization*, such that

1. $\mathcal{Q}$ maps a function F to an operator $\widehat{F} = \mathcal{Q}(F)$, i.e., classic observables to quantum ones.

2. $\mathcal{Q}$ has an algebraic property $\widehat{\{F_1, F_2\}} = [\widehat{F}_1, \widehat{F}_2]$, i.e., maps the Poisson brackets to the commutators.

3. $\mathcal{Q}(1) = I$, i.e., the image of the function 1 identically equal to one is the identity operator I.

Unfortunately, $\mathcal{Q}$ does not exists even for very simple cases. For example, the "no go" theorem of VAN HOVE in [17] states that the canonical commutation relations (4.1) could not be extended even for polynomials in X and Y of degree greater than 2.

As a solution one could try to select a primary set of classic observables F_1, ..., F_n, which defines coordinates on M and thus any other observable F is a function of F_1, ..., F_n: $F = F(F_1, \ldots, F_n)$. For such a small set of observables a quantization $\mathcal{Q}$ always exists (for example the *geometric quantization* [24, § 15.4]) and one could hope to construct general quantum observable $\widehat{F}$ as a function of the primary ones $\widehat{F} = F(\widehat{F}_1, \ldots, \widehat{F}_n)$.

But to do that one should answer the question equivalent to functional calculus: *What is a function of n operators?* For a commuting set of operators as well as for the case $n = 1$ it is easy to give an answer in terms of *algebra homomorphisms*: the functional calculus Φ is an algebra homomorphism from an algebra of function to an algebra of operators, which is fixed by a condition $\Phi(F_i) = \widehat{F}_i$, $i = 1, \ldots, n$. But for physically interesting case of several non-commuting $\widehat{F}_i$ this way is impossible.

Several solutions were given whether analytically by integral formulas like for the Feynman [16] and Weyl [2] calculi or algebraically by "ordering rules"

like for the Wick [43, § 6] and anti-Wick [5] calculi. These approaches are connected: if calculus is defined via integral formula then it correspond to some ordering (e.g., the Feynman case) or symmetrization (e.g., the Weyl case); and *vise versa* starting from particular ordering a suitable integral representation could be found [5].

We will consider the Weyl functional calculus in more details. Let $\mathbf{T} = \{T_j\}$, $j = 1, \ldots, n$ be a set of bounded selfadjoint operators. Then for every set $\{\xi_j\}$ of real numbers the unitary operator $\exp(i \sum_1^n \xi_j T_j)$ is well defined. The *Weyl functional calculus* [2] is defined by the formula

$$f(\mathbf{T}) = \int_{\mathbb{R}^n} \exp\left(i \sum_1^n \xi_j T_j\right) \widehat{f}(\xi) \, d\xi, \tag{8.3}$$

where $\widehat{f}(\xi)$ is usual Fourier transform of $f(x)$. In [2] the following properties of the Weyl calculus were particularly proved:

1. The image of a polynomial $p(x)$ is the symmetric in T_j polynomial $p(\mathbf{T})$ (e.g. image of $p(x_1, x_2) = x_1 x_2$ is $p(T_1, T_2) = \frac{1}{2}(T_1 T_2 + T_2 T_1)$).

2. The Weyl calculus commutes with affine transformations, i.e., let $S_i = \sum_{j=1}^n m_{ij} T_j$ for a nonsingular matrix $\{m_{ij}\}$, then

$$f(\mathbf{S}) = g(\mathbf{T}) \text{ where } g(x) = f(m \cdot x). \tag{8.4}$$

Moreover the last condition together with the requirement that Weyl functional calculus gives the standard functional calculus for polynomials in one variable completely defines it. We know that functional calculus for one variable could be also defined by its covariance property. Thus we could give an equivalent definition[12]

Definition 8.1 The Weyl functional calculus for an n-tuple $\mathbf{T}$ of selfadjoint operators is a linear map from a space of functions on $\mathbb{R}^n$ to operators in a C^*-algebra $\mathfrak{A}$ such that

1. Intertwines two representation of affine transformation, namely (8.4) holds.

2. Maps the function e^{ix_1} on $\mathbb{R}$ to operator

$$e^{iT_1} = \sum \frac{(iT_1)^n}{n!}. \tag{8.5}$$

---

[12]We skip details because the problem is beyond the L_2 scope presented in the paper.

While the affine group is natural for Euclidean space one could need other groups under different environments. For example, one could consider a non-commutative geometry generated by the group of conformal (or Möbius) transformations. It is known [9] that the group M of Möbius (mapping spheres onto spheres) transformations of $\mathbb{R}^n$ could be represented via fraction-linear transformations with the help of Clifford algebras $\mathcal{C\!l}(n)$. Taking the tensor product $\widetilde{\mathfrak{A}} = \mathfrak{A} \otimes \mathcal{C\!l}(n)$ of an operator algebra $\mathfrak{A}$ and Clifford algebra we could consider a representation of conformal group M in the non-commutative space $\widetilde{\mathfrak{A}}$, which also is defined by fraction-linear transformations.

Such an approach leads particularly to a natural definition of joint spectrum of n-tuples of non-commuting operators and a functional calculus of them [26]. The functional calculus is not an algebra homomorphism (for $n > 1$) but is an intertwining operator between two representations of conformal group M. Nevertheless the joint spectrum obeys a version of the spectral mapping theorem [26].

It is naturally to expect that physically interesting functional calculi from group representations are not restricted to two given examples. Looking on these functional calculi as on a quantization one found that common contradiction between the group covariance and an algebra homomorphism property vanishes (as the algebra homomorphism property itself). It turns that the group covariance alone is sufficient for the definition of functional calculus. Moreover such an approach gives a unified description of all known functional calculi and suggests new ones.

One should be aware that algebraic structure it still important: it enters to the definition of the group action via fraction-linear transformations (7.2) or definition of exponent (8.5). So the question posed at the title of Section might have an answer: *Yes, we need that observables form an algebra, but we do not need that quantization will be connected with an algebra homomorphism property in general.*

9   Conclusion

> We often hear that mathematics consists mainly in "proving theorems". Is a writer's job mainly that of "writing sentences"?
>
> G.-C. Rota [21, Chap. 11].

Looking through the present discussion one could not resist to the following feeling:

The goal of new mathematical facts is to elaborate the true language for an understanding old ideas, such that they will be expressed with the ultimate

perfectness.

This goal could not be obviously accomplished within a finite time. So we have a good reason for self-irony mentioned at the end of Introduction.

10   Acknowledgments

In the paper we try to establish some connections between several areas of algebra, analysis, and geometry. A reasonable bibliography to such a paper should be necessary incomplete and the given one is probably even not representative. Any suggestions about unpardonable omitted references will be thankfully accepted.

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